

# Auction Theory With Experiments

Rod Garratt<sup>1</sup>

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<sup>1</sup>Department of Economics, University of California at Santa Barbara, garratt@ucsb.edu.



# Overview

The goal of this text, its supplement, and software is to provide a means of teaching auction theory to undergraduates. The starting premise of this text is that auction theory is intrinsically interesting and practical but mathematically difficult. As a result, it is necessary to (1) find the simplest presentation of the material that still conveys the full power of the results and (2) suitably motivate students to work hard and comprehend the material. We achieve the first task by presenting most of the theoretical results in terms of a special case: symmetric, uniformly distributed use-values with common support. This case allows simple, closed form solutions of bid functions and easy computation of auction revenues. Moreover, this case allows us to demonstrate central results, such as revenue equivalence of standard auction formats, with relative ease. We achieve the second task by using experiments in a completely integrated way. Each topic covered in this text relates to an experiment. Standard theoretical results covered in the text are presented as predictions for behavior in these experiments. The supplement outlines data analysis to test the theory for each experiment and provides discussion points. Sometimes, the discussion involves analysis provided by the students in homework exercises that accompany some of the chapters. A list of the experiments used in this text is provided in the Appendix.

## Target Audience

The text uses basic concepts from calculus and probability theory. Ideally, students who take this course will have taken a course in calculus, at least at the high school level. A review of the needed concepts is provided in Chapter 2, which is based on a section called Probability Preliminaries, in the unpublished working paper “A Technical Primer on Auction Theory I: Independent Private Values” by Steven Matthews.

Most of the theory presented in the text is game theoretic. As such, it is preferable that students who enroll in this course will have already completed an undergraduate course in game theory. However, we understand that this will not always be possible. All we really require is that students understand the notion of a dominant strategy and have a working knowledge of Nash equilibrium. Most intermediate microeconomics courses include one or two chapters on game theory that cover these topics. We regard this to be sufficient.

The course includes empirical analysis of experimental data; however students are not required to have taken econometrics.



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Getting Started</b>	<b>3</b>
2.1	Order Statistics . . . . .	4
2.2	Conditional Expectations . . . . .	6
<b>3</b>	<b>Bidding Behavior</b>	<b>9</b>
3.1	Second-Price Auctions . . . . .	9
3.1.1	Reserve Prices and Entry Fees . . . . .	11
3.1.2	Sunk Cost Fallacy . . . . .	11
3.2	First-Price Auctions . . . . .	12
3.2.1	Reserve Prices and Entry Fees . . . . .	14
3.2.2	Risk Aversion . . . . .	15
<b>4</b>	<b>Revenue</b>	<b>19</b>
4.1	Expected Revenue . . . . .	19
4.2	Revenue Equivalence . . . . .	20
4.3	Revenue With a Reserve Price or Entry Fee . . . . .	21
<b>5</b>	<b>Optimal Auctions</b>	<b>23</b>
5.1	Optimal Reserve with $n \geq 1$ Bidders . . . . .	24
5.2	Saliency of the Reserve Price Decision . . . . .	26
<b>6</b>	<b>Common Value Auctions</b>	<b>31</b>
6.1	Model I . . . . .	33
6.2	Model II . . . . .	37
6.2.1	Revenue Equivalence . . . . .	39



# Chapter 1

## Introduction

Auctions have been used to sell goods for thousands of years. The earliest recorded examples are the Babylonian wedding auctions which took place around 500 BC. These auctions were oral, descending price auctions in which the seller started off by proposing an unacceptably high price for the young bride-to-be and then lowered this price, incrementally, until someone accepted it. In some cases, brides went for a negative price, the equivalent of a dowry, paid out of the profits from auctions that terminated at a positive price.<sup>1</sup>

So why did the Babylonians use this method? Or more generally, why does anyone use an auction to sell an item rather than simply offering the item for sale at a particular price? The answer is quite simple: auctions are used when the seller does not know what price to charge. This means that first of all, the item must be somewhat unique: there is not a market price determined by forces of supply and demand, as would be the case if the item were a bushel of wheat or a gallon of milk. And second of all, the item must have either unknown common value or varying private value.

Unknown common value means that the item is worth the same to whomever obtains it, but no one is exactly sure what that value is. Therefore different people will be willing to pay different amounts based on their best guess of the value. An example of an item with unknown common value is a tract of land that the government auctions off to potential bidders for oil exploration. Each interested party may be allowed to sink delineation wells that provide a prediction about the true content of the field. However, none of the oil companies that take place in the auction know exactly how much oil can be extracted from the land or what the future price of oil will be.

Varying private values relate to the fact that people have different tastes and incomes and hence differ in their willingness and ability to pay for the same object. Both willingness and ability to pay matter for obvious reasons. Regardless of how much a person might say they like or want an item, their valuation, which we take to mean the maximum amount they would agree to pay for the item, can be no higher than the amount of money they have available to spend on it. Pieces of art are good examples of commodities with varying private values. So are antiques and classic cars, which are valued as collectibles. These items are commonly sold at auction, in some cases at high profile, established auction houses like Christies (1766) and Sothebys (1804) that have operated for hundreds of years.

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<sup>1</sup>Details of these auctions are due to the historian Herodotus.

At the most fundamental level, auctions are a means of price discovery. The seller may know what the item is worth to her and she may insist, at the very least, that the auction cannot be completed at a price lower than that. However, if she simply announced a price, the item may go unsold. Or she may find that there are multiple willing buyers, evidence that the announced price was too low. The seller often wants to get the highest price she can from the pool of interested buyers. However, each of these potential buyers may value the item differently, and none have an incentive to reveal their valuation to the seller.

The tension between the seller's desire to maximize the revenue from the auction and each bidder's desire to obtain the object at the lowest possible price is the crux of auction theory. To understand the different incentives and tradeoffs faced by sellers and bidders it is necessary to think about auctions as a game. The seller is playing against the buyers. Her action in the game is the choice of an auction mechanism. I have already suggested that the seller's objective in choosing a mechanism might be to maximize revenue, but there are certainly other interesting objectives that we should and will consider. For instance, the primary goal of an auction may be allocative efficiency, putting the object in the hands of the person that values it most, while revenue is only a secondary consideration.<sup>2</sup> Meanwhile, the potential buyers are playing against the seller and each other.

We will often focus on the subgame between the potential buyers, which takes place after the auction mechanism has been decided. We will consider a wide range of auction mechanisms, some of which will be familiar to you and some of which will not. Examples of these different auction mechanisms are seen throughout the world. Tulips in Holland are sold by a descending price auction. United States government sales of timber and off-shore oil are often carried out by first-price sealed bid auctions. Antiques and art are typically sold by oral ascending auctions. What we learn about these auctions, both in theory and practice, will help us to understand why these different auction formats were chosen in the first place.

There has been a stunning increase in the popularity of auctions in the last twenty years. This is in large part due to the emergence of the internet. Online auction houses such as eBay in the United States and Yahoo! in Japan give sellers access to huge numbers of potential buyers. However, they also introduce competition between sellers. Items which may have been perceived as unique in small community auctions are now offered alongside other similar items from sellers residing thousands of miles away. These new developments present interesting challenges for the auction theorist and practitioners like yourself. We will tackle some of the issues unique to online auctions, such as the role of feedback profiles, hidden reserves and second-chance offers. In the process, we will increase your understanding of these options and quite possibly help you to take advantage of these features when you participate in online auctions yourself!

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<sup>2</sup>This was said to be true in a speech by then Vice President Al Gore at the opening of the December 1994 Broadband PCS spectrum auctions.

# Chapter 2

## Getting Started

We seek a theory of bidding behavior in auctions. Our theory will attempt to explain how peoples' bids are related to their individual valuations, or simply values, for the item being auctioned. In mathematical terminology, we want to determine a mapping from values to bids. A person's value is the hypothetical price at which she is indifferent between buying the item and not buying it. For now, we do not consider the possibility that the person might resell the item. Hence, value reflects only the consumer's willingness to pay to consume the item. With this interpretation, the payoff someone gets from winning an item in an auction is the difference between her value and the actual price she pays for it, which depends on the bids and the auction rules.

We cannot presume to know peoples' values for items they bid on in auctions. An individual's willingness to pay for something depends on how much satisfaction or enjoyment she gets from consuming it instead of other things she could potentially consume in its place. These feelings are unobservable. Moreover, a person's ability to pay for something depends on their wealth or disposable income and these amounts are also often private information to the bidder. However, we can't proceed unless we know something about individual values. Hence, we make an assumption about the distribution of values that can be attributed to individuals in society. That is, we assume that we know the range of possible values that people hold and the likelihood that someone's value lies in any interval.

The knowledge we will assume to have regarding how likely it is that a person has a value in a particular range will be summarized by a probability distribution function  $F$ . For any (normalized) value  $v$ , the number  $F(v)$  tells us the probability that an individual has a value that is less than or equal to  $v$ . For example, suppose that value lies between 0 and \$100 and  $F(50) = .75$ . This says that the probability that an individual's value is less than or equal to \$50 is .75. In this case, low values (values in the lower half of the possible range of values) are more likely than high values. The probability that a person will have a value above \$50, the midpoint of the value range, is only  $1 - F(50) = .25$ .

You can think of the function  $F$  as representing the distribution of types of people that exist in the population. Then,  $F(v)$  represents the fraction of the population with types less than  $v$ . When an individual shows up at an auction we can think of her as being a random draw from the population. Hence, the likelihood that she has a value in any range is simply given by the fraction of the population that has values in that range.

It is often convenient to assume that the lower bound on peoples' values is 0. When values

range from 0 to some maximum value  $v_{\max}$ , we require that  $F(0) = 0$  and  $F(v_{\max}) = 1$ . Otherwise,  $F$  is not a probability distribution function. A convenient special case arises when  $F(v) = v/v_{\max}$  for any value  $v$  between 0 and  $v_{\max}$ . Then we say values are *uniformly distributed* on  $[0, v_{\max}]$ . This means that all value types between 0 and  $v_{\max}$  are equally represented in the population, and hence the probability of seeing an individual at an auction with a value in the first one-third of the value range is  $1/3$ , in the first half of the value range is  $1/2$ , etc. The probability of observing a value in any interval is simply equal to the ratio of the length of that interval and the length of the value range. For instance, if the value range is 0 to \$100, the probability of drawing a value between \$30 and \$70 is  $(70 - 30)/100 = .4$ . This is just the difference between the probabilities associated with the upper and lower bounds of the interval.

It is important to really think about what  $F$  is and how to interpret it.  $F(v)$  is *not* the probability of observing the value  $v$ . In the framework outlined here values are defined on a continuous range. Hence, there are so many possible values (an uncountably infinite number in fact) that the probability of observing any one particular value is zero! That is why we think in terms of the likelihood of observing values below a certain amount or in a particular range. Our model is a good approximation to real world situations where the upper bound on values is reasonably large and prices can be denominated in dollars and cents. If values are uniformly distributed between 0 and \$10,000, for example, the chance that an individual has a value of, say, \$4,324.35 is one in a million, which is approximately 0.

The preceding description of values is formalized as follows:

- Bidder  $i$ 's value is denoted  $v_i$  and is assumed to lie in the interval  $[0, v_{\max}]$ .
- Use  $\sim$  to denote a random variable.
- $\tilde{v}_i$  is a random variable whose realization  $v_i$  is the use value of bidder  $i$ .
- $F(\cdot)$  is the probability distribution function of  $\tilde{v}_i$ ;  $F(v) = \Pr[\tilde{v}_i \leq v]$ .

We make the following assumptions about the probability distribution function  $F$ .

**Assumption 1.**  $F(\cdot)$  is a non-decreasing function on  $[0, v_{\max}]$ .

**Assumption 2.**  $F(0) = 0$  and  $F(v_{\max}) = 1$ .

**Assumption 3.**  $F(\cdot)$  has a continuous and positive derivative on  $[0, v_{\max}]$ ,  $f(v) = F'(v)$ .

Assumptions 1 and 2 are required assumptions for a probability distribution function. Assumption 3 is for convenience. We call  $f(\cdot)$  the density function of the random variable  $\tilde{v}_i$ . Note that since  $F(\cdot)$  has a continuous and positive derivative on  $[0, v_{\max}]$ , no individual value in the interval  $[0, v_{\max}]$  occurs with strictly positive probability, and hence,  $\Pr[\tilde{v}_i < v] = \Pr[\tilde{v}_i \leq v] = F(v)$ . This means we do not have to distinguish between outcomes that are “less than” some number and outcomes that are “less than or equal to” some number.

## 2.1 Order Statistics

Our predictions for auction outcomes will depend on what we expect bidders' values to be. We don't need to predict all bidders' values. What usually matters are the highest or second-

highest values. These are what typically determine who wins an auction or how much they pay.

In an auction with  $n$  bidders, the highest and second-highest values are called the  $n$ th and  $(n - 1)$ th order statistics, respectively. Since the vector of values that bidders possess is not known to us, we must treat each of these order statistics as a random variable. Given our knowledge of the probability distribution function  $F$  for individual values we can compute probability distribution functions for these random variables. Let  $\tilde{v}^{(1)}$  and  $\tilde{v}^{(2)}$  denote the random variables that correspond to the highest and second-highest elements of  $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ , the set of random variables that represent all the individual values. We want the probability distribution functions for  $\tilde{v}^{(1)}$  and  $\tilde{v}^{(2)}$ .

First let's consider the random variable  $\tilde{v}^{(1)}$ . In order for  $\tilde{v}^{(1)}$  to be less than  $v$ , it must be the case that each of the individual values is less than  $v$ . That is, we require  $\tilde{v}_1 < v$ ,  $\tilde{v}_2 < v$ ,  $\tilde{v}_3 < v$ , etc. all the way up to  $\tilde{v}_n < v$ . The probability that any  $\tilde{v}_i < v$  is  $F(v)$ . That is simply the definition of  $F(v)$ . Since we assume the  $n$  bidders' use values are independently distributed, the probability that the highest of the  $n$  random variables,  $\tilde{v}_1$  through  $\tilde{v}_n$ , is less than or equal to  $v$  is the product of the  $n$  probabilities that each one of them is less than  $v$ . That is,

$$F^{(1)}(v) = \Pr[\tilde{v}^{(1)} \leq v] = \prod_{i=1}^n \Pr[\tilde{v}_i \leq v] = F(v)^n.$$

To find the density function of  $\tilde{v}^{(1)}$ , simply take the derivative,

$$f^{(1)}(v) = F'^{(1)}(v) = nF(v)^{n-1}f(v).$$

**Example 2.1.** Suppose the random variable  $\tilde{v}_i$  is uniformly distributed on the interval  $[0, 100]$ . Then  $F^{(1)}(v) = \left(\frac{v}{100}\right)^n$  and  $f^{(1)}(v) = \frac{nv^{n-1}}{100^n}$ .

Now let's consider the random variable  $\tilde{v}^{(2)}$ . There are a number of ways that the realization of the random variable  $\tilde{v}^{(2)}$  can be less than  $v$ . We already considered one of them. Namely, each of the individual values is less than  $v$ . This occurs with probability  $F(v)^n$ . Another way for the realization of the random variable  $\tilde{v}^{(2)}$  to be less than  $v$  is for  $\tilde{v}_1$  to be greater than  $v$  and for each of the other random variables,  $\tilde{v}_2$  through  $\tilde{v}_n$ , to be less than  $v$ . The probability that  $\tilde{v}_1$  is greater than  $v$  is  $1 - F(v)$ . This is simply one minus the probability that  $\tilde{v}_1$  is less than  $v$ . The probability that all of the other  $n - 1$  random variables is less than  $v$  is  $F(v)^{n-1}$ . This is simply the probability that one of the values is less than  $v$ , which is  $F(v)$ , multiplied by itself  $n - 1$  times. Combined, the probability that  $\tilde{v}_1$  is greater than  $v$  and all the rest are less than  $v$  is  $(1 - F(v))F(v)^{n-1}$ . But, this can happen  $n$  different ways. Instead of  $\tilde{v}_1$  greater than  $v$  and all the rest less than  $v$ , we could have  $\tilde{v}_2$  greater than  $v$  and all the rest less than  $v$ , etc. In other words, any one of the  $n$  bidders can be the one whose value is greater than  $v$ . Hence, the probability that (any) one value is greater than  $v$  and all the rest are less than  $v$  is  $n(1 - F(v))F(v)^{n-1}$ . There is no other way for the second-highest value to be less than  $v$ . Hence, the probability that the second-highest of the  $n$  random variables  $\tilde{v}_i$ ,  $i = 1, \dots, n$ , is less than or equal to  $v$  is

$$\begin{aligned} F^{(2)}(v) &= \Pr[\tilde{v}^{(2)} \leq v] = F(v)^n + n[(1 - F(v))F(v)^{n-1}] \\ &= F(v)^n + n[F(v)^{n-1} - F(v)^n]. \end{aligned}$$

To find the density function of  $\tilde{v}^{(2)}$ , simply take the derivative,

$$f^{(2)}(v) = n(n-1)F(v)^{n-2}[1-F(v)]f(v).$$

**Example 2.2.** Suppose the random variable  $\tilde{v}_i$  is uniformly distributed on the interval  $[0, 100]$ . Then

$$F^{(2)}(v) = \left(\frac{v}{100}\right)^n + n \left[ \left(\frac{v}{100}\right)^{n-1} - \left(\frac{v}{100}\right)^n \right] = n \left(\frac{v}{100}\right)^{n-1} - (n-1) \left(\frac{v}{100}\right)^n$$

and

$$f^{(2)}(v) = n(n-1) \left( \frac{v^{n-2}}{100^{n-1}} - \frac{v^{n-1}}{100^n} \right).$$

In addition, we will sometimes use the random variable  $\tilde{y} = \max\{\tilde{v}_1, \dots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \dots, \tilde{v}_n\}$ . This is the maximum of the values of all the bidders excluding bidder  $i$ . Since we assume that all bidders have values determined by the same distribution  $F$ , bidder  $i$  can be any one of the bidders (including bidder 1 or bidder  $n$ ). Let  $G(\cdot)$  denote the distribution function for the random variable  $\tilde{y}$ . Note that

$$G(y) = F(y)^{n-1} \quad \text{and} \quad g(y) = (n-1)F(y)^{n-2}f(y).$$

**Example 2.3.** Suppose the random variable  $\tilde{v}_i$  is uniformly distributed on the interval  $[0, 100]$ . Then  $G(y) = \left(\frac{y}{100}\right)^{n-1}$  and  $g(y) = (n-1)\frac{y^{n-2}}{100^{n-1}}$ .

## 2.2 Conditional Expectations

In many instances we will want to compute the expected value of a random variable when we have additional information about its range of possible outcomes. For example, suppose we wanted to compute the expected value of the losing bidder's value in a two-bidder auction. Furthermore, suppose the winning bid was  $b_1$ . Then we can use information about the equilibrium bid function and the value of the winning bid to limit the range of possible values of the losing bidder's value. In the case of a second price auction, where bidders bid their value in equilibrium, we can assume the loser's value is less than  $b_1$ . In the case of a first-price auction with  $F(\cdot)$  uniform, where each of the two bidders bid  $1/2$  times their value in equilibrium, we can assume that the loser's value is less than  $2b_1$ .

For now, let's not stick with any one auction format, but rather let's assume that our information about the auction tells us that the loser's bid is less than some amount  $k$ . In order to compute the expected value of the loser's bid we need to compute the expected value of the random variable  $\tilde{v}$  conditional on the knowledge that the maximum possible realization  $v$  is less than  $k$ . To do this, we need to know its density function. We start with its distribution function, which we will call  $L$ . The distribution function  $L$  has the same properties as the distribution function  $F$ , except that it has to be normalized so that  $L(k) = 1$ . These facts imply

$$L(v) = \frac{F(v)}{F(k)}.$$

The density function is  $L'(v) = l(v) = \frac{f(v)}{F(k)}$ . Hence, the expected value of the random variable  $\tilde{v}$ , conditional on its realizations  $v < k$ , is

$$\int_0^k v \frac{f(v)}{F(k)} dv.$$

**Example 2.4.** Assume that bidders' values are distributed uniformly on  $[0, 100]$  and consider  $k \leq 100$ . It is immediate that our required density function is  $l(v) = \frac{1}{100}$ . Hence, the expected value of the uniformly distributed, random variable  $\tilde{v}$ , conditional on its realizations  $v < k$ , is

$$\int_0^k v \frac{1}{100} dv = \frac{1}{100} \left[ \frac{v^2}{2} \right]_0^k = \frac{k^2}{200}.$$

The result for the uniform distribution is pretty intuitive when you think about it. All values between 0 and  $k$  are assumed to be equally likely, so the expected value is  $\frac{k}{2}$ . Sticking with the uniform distribution we can compute the conditional expectations discussed above.

**Example 2.5.** In the case of a 2-bidder first-price auction with uniform  $F$  and winning bid  $b_1$ , the expected value of the losing bidder's value is  $b_1$ .

**Example 2.6.** In the case of a 2-bidder second-price auction with uniform  $F$  and winning bid  $b_1$ , the expected value of the losing bidder's value is  $\frac{b_1}{2}$ .



# Chapter 3

## Bidding Behavior

The material in this chapter is motivated by Experiments 1–8. Instructions for these experiments are available in the appendix.

We assume people attempt to maximize their payoff from participating in an auction. Hence, we are in a sense trying to determine their optimal bids. However, an auction is a game in which the payoff an individual earns from any given bid depends on the bids placed by others. Hence, the notion of an optimal or “best” bid cannot usually be defined in isolation. We need some conjecture about how others will bid.

Our prediction for bidding behavior is based on the idea that people bid optimally based on their predictions about how others will bid and that these predictions are correct. How everyone bids depends, of course, on their values. It is assumed that each bidder knows her own value, but only knows the probability distribution function that determines other bidders’ values. Hence, bidders must best respond to their beliefs regarding the bid functions used by other bidders and the likelihood of the values that will enter into these bid functions. In other words, our predictions for bidding behavior are the Bayes Nash equilibrium bid functions. A *Bayes Nash equilibrium* for an auction is a bid-function profile  $b = (b_1(\cdot), \dots, b_n(\cdot))$  such that for each bidder  $i$  and each possible value  $v_i$  for bidder  $i$ , the bid  $b_i(v_i)$  maximizes bidder  $i$ ’s expected payoff given the vector  $b_{-i} = (b_1(\cdot), \dots, b_{i-1}(\cdot), b_{i+1}(\cdot), \dots, b_n(\cdot))$  of bid functions for the other  $n - 1$  bidders. This definition is not quite operational yet, since we do not explain how to compute the expected payoff to a bid. We will do this explicitly in the following sections, using the particulars of the different auction formats.

We start off by considering the second price auction. You may wonder why we don’t start with the first-price auction. The answer is that the theoretical predictions for the second-price auction are much easier to derive. This doesn’t mean that second-price auctions are less interesting, however. Reconciling observed behavior with the theory can be equally challenging, with just as many opportunities to gain insight into how people perceive auctions.

### 3.1 Second-Price Auctions

In the second-price auction the bidder who places the highest bid wins and she pays the amount of the second-highest bid. In these auctions, your bid determines whether or not

you are the auction winner, but it does not determine how much you pay. In the event you are the high bidder and win the auction you pay the amount of the second-highest bid. You might think that this suggests you should bid very high to ensure that you win the auction. And, in experiments this occurs quite often.<sup>1</sup> However, it is important to understand that whenever you bid above your value there is a chance that you will win the auction and pay a price that is above your value. That is, you might lose money! This happens when the second-highest bid is below your bid, but above your value.

Let's consider a 2-bidder auction where values range from 0 to \$100. Suppose, for example, that your value is \$52, you bid \$90, and the only other player bids \$70. In this case, because you have the highest bid you win the auction. However, your earnings from the auction would be  $\$52 - \$70 = -\$18$ . That is, you lose \$18. In fact, whenever you bid above your value you run the risk of losing money in this way. Moreover, it is important to understand that you cannot improve your payoff by bidding more than your value! If you need to bid more than your value to win the auction, it means the price you will have to pay is more than your value, in which case you will lose money by winning the auction. If the second-highest bid is below your value, then a bid equal to your value will win the auction anyway and earn the same payoff as a higher bid.

It is also true that you cannot increase your payoff by bidding less than your value and, in fact, you might miss out on possible gains. Suppose your value is \$52, you bid \$35, and the only other player bids \$40. In this case, because you do not have the highest bid you do not win the auction. Your earnings from the auction would be 0. Any bid above \$40 would have caused you to win the auction and earned you a payoff of  $\$52 - \$40 = \$12$ . It does not cost you anything to bid as high as your value and in fact you might gain by doing so if the other player's bid is higher than \$40 but below \$52. If you bid above your value, however, you run the risk of losing money if the other player bids more than your value.

**Theorem 3.1.** *The unique equilibrium in weakly-dominant strategies of the second-price auction consists of each bidder  $i$  following the strategy*

$$b^H(v_i) = v_i.$$

The reasoning behind this result was provided in the preceding text for the case of two bidders. Here is a formal argument that applies to any number of bidders.

*Proof.* Let  $b_i$  denote the bid of bidder  $i$  and let  $B$  denote the highest bid among everyone else. Then we can formally write the payoff to bidder  $i$  from each possible bid  $b_i$ , as a function of  $B$  and bidder  $i$ 's value  $v_i$ . If  $b_i < B$  bidder  $i$  does not win the auction and her payoff is 0. If  $b_i = B$  then there is a tie between bidder  $i$  and all of the other bidders that bid  $B$ . If we assume that the number of other bidders that bid  $B$  is  $K$  and that one of the tied bidders including bidder  $i$  will be chosen as the winner randomly with equal probability, then bidder  $i$ 's payoff is  $(v_i - B)/(K + 1)$ . Finally if  $b_i > B$ , then bidder  $i$  wins the auction alone and receives the payoff  $v_i - B$ .

Now consider three distinct and exhaustive cases that describe possible relationships between  $v_i$  and  $B$ . If  $v_i > B$ , then any bid greater than  $B$  earns the maximal payoff of

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<sup>1</sup>See Kagel and Levin, 1993.

$v_i - B$ . If  $v_i < B$ , then any bid less than  $B$  earns the maximal payoff of 0. Finally, if  $v_i = B$ , then all bids earn the same payoff, namely, 0. In all three cases, bidder  $i$ 's payoff is maximized by the bid  $b_i = v_i$ .  $\square$

Other asymmetric equilibria in weakly dominated strategies exist. Suppose the highest use value is \$100. Then,  $b_1 = \$100, b_2, \dots, b_n = 0$  is also an equilibrium. However,  $b_1 = \$100$  is weakly dominated by  $b_1 = v_1$  and  $b_i = 0$  is weakly dominated by  $b_i = v_i$ , for  $i = 2, \dots, n$ . See Blume and Heidhues (JET, 2005).

*In-class discussion of Experiments 1 and 2: SPA with no entry fee and no reserve for 2- and 5-bidder auctions.*

### 3.1.1 Reserve Prices and Entry Fees

More generally, assume that the second-price auction has an entry fee,  $c$ , and a reserve,  $r$ , where (in terms of normalized values)  $0 \leq c \leq 100 - r \leq 100$ . Since the entry fee is paid regardless of whether or not a bidder wins the auction, it is no longer a weakly dominant strategy for bidders to bid their value. In particular, bidders with low values will choose not to participate in the auction because it is unlikely they will earn enough profit in the auction to cover the entry fee (or even impossible if their value is less than  $c$ ). The implication is that only bidders with high enough values will enter the auction. How high is high enough? Let's think about an equilibrium in which only bidders with a value above a certain cutoff,  $v_0$ , choose to participate. As always, we let  $\tilde{y}$  denote the random variable whose realization  $y$  is the maximum of  $n - 1$  independent draws from  $F$ . Recall that the distribution function of  $\tilde{y}$  is  $G(y)$ . The payoff to a bidder of type  $v_0$  to participating and bidding  $v_0$  is  $(v_0 - r)G(v_0)$ . In order for this bidder to be indifferent between participating and not participating we require

$$(v_0 - r)G(v_0) = c.$$

This equation defines our critical value  $v_0$ . A bidder with a value greater than  $v_0$  has a payoff at least as large as this, because she still wins in the scenarios where  $y$  is less than  $v_0$ , and now she also wins in scenarios where  $y$  is greater than  $v_0$  but less than her value. Hence any bidder with a value greater than  $v_0$  will participate in the auction. Given that all bidders with values greater than  $v_0$  participate we can use the same reasoning as we did in the case of no entry fee to show that value-bidding is an equilibrium. Thus, we have established that an equilibrium of second-price auction with reserve  $r$  and entry fee  $c$  consists of each bidder following the strategy

$$b^{II}(v_i) = \begin{cases} v_i & \text{if } v_i \geq v_0 \\ \text{No} & \text{if } v_i < v_0 \end{cases}$$

where  $v_0$  is the solution to  $(v_0 - r)G(v_0) = c$ .

This symmetric equilibrium is in weakly dominant strategies only if  $c = 0$ .

### 3.1.2 Sunk Cost Fallacy

The theory predicts that once you have paid the entry fee you should bid the same way as you would if you had not entered. This is not surprising if you think about it. The

argument we gave to show that value bidding is a weakly dominant strategy does not depend on the support of values for other bidders. Yet, it has been demonstrated in laboratory experiments that subjects often bid differently after paying an entry fee. In particular, experiment evidence reported in Charness, Garratt and Hartman (2008) shows that people often deduct a fraction of their entry fee from their bid. Once a subject has paid to enter the auction, any entry fee that they might have paid should have no bearing on their bidding decision. This is a nice instance of the sunk cost fallacy.

*In-class discussion of Experiments 3 and 4: 2-bidder SPA with no entry fee and reserve  $r = \$50$ , 2-bidder SPA with entry fee  $c = \$25$  and no reserve.*

## 3.2 First-Price Auctions

In the first-price auction the bidder who places the highest bid wins and she pays the amount of her bid. In these auctions, your bid determines whether or not you are the auction winner and the price you pay. Since you pay your own bid it is not optimal to bid your value, as was the case for the second-price auction. This would ensure you a payoff of 0. By lowering your bid below your value you make a positive payoff when you do win. However, lowering your bid reduces the probability that you win the auction. In order to win the auction your bid must be higher than everyone else's. Assuming other bidders also bid some fraction of their values, their bids will be random variables. Hence, the lower your bid the lower the chance that all bids are below yours. The optimum bid must therefore balance the gain in payoff from bidding lower, and hence earning a higher profit when you win the auction, with the loss in probability of winning.

**Theorem 3.2.** *The unique, symmetric equilibrium of the first-price auction consists of each bidder following the strategy*

$$b^I(v_i) = \int_0^{v_i} y \frac{g(y)}{G(v_i)} dy$$

The following proof allows bidders to have any probability distribution function over values that satisfy our three assumptions in Chapter 2. This proof can be skipped by most readers. However, everyone should attempt to understand the arguments for the case of uniform  $F$  that follow this proof.

*Proof.* Note that any bid greater than  $b^I(1)$  is dominated by  $b^I(1)$ . The payoff to a single deviator from bidding  $b^I(z)$ ,  $z \in [0, 1]$ , when value is  $v$  (ignore subscript  $i$ ) is

$$\begin{aligned} \Pi(b^I(z), v) &= G(z)[v - b^I(z)] \\ &= G(z)v - \int_0^z yg(y) dy \\ &\quad \text{(integration by parts)} \\ &= G(z)v - G(z)z + \int_0^z G(y) dy \\ &= G(z)(v - z) + \int_0^z G(y) dy. \end{aligned}$$

Likewise,

$$\begin{aligned}
 \Pi(b^I(v), v) &= G(v)[v - b^I(v)] \\
 &= G(v)v - \int_0^v yg(y) dy \\
 &\quad \text{(integration by parts)} \\
 &= G(v)v - G(v)v + \int_0^v G(y) dy \\
 &= \int_0^v G(y) dy.
 \end{aligned}$$

Hence, the loss to deviating is

$$\Pi(b^I(v), v) - \Pi(b^I(z), v) = \int_0^v G(y) dy - G(z)(v - z) - \int_0^z G(y) dy. \quad (*)$$

Consider  $z > v$ :

$$\begin{aligned}
 (*) &= G(z)(z - v) - \int_z^v G(y) dy \\
 &= \int_z^v G(z) dy - \int_z^v G(y) dy \\
 &= \int_z^v [G(z) - G(y)] dy > 0.
 \end{aligned}$$

Consider  $z < v$ :

$$\begin{aligned}
 (*) &= \int_v^z G(y) dy - G(z)(v - z) \\
 &= \int_v^z [G(y) - G(z)] dy > 0.
 \end{aligned}$$

Hence, deviating from  $b^I(v)$  is always costly. □

In the case of  $F$  uniform, the equilibrium bid function is

$$\begin{aligned}
 b^I(v_i) &= \int_0^{v_i} y \frac{(n-1)y^{n-2}}{v_i^{n-1}} dy \\
 &= \frac{n-1}{v_i^{n-1}} \int_0^{v_i} y^{n-1} dy \\
 &= \frac{n-1}{v_i^{n-1}} \left[ \frac{y^n}{n} \right]_0^{v_i} \\
 &= \frac{n-1}{v_i^{n-1}} \cdot \frac{v_i^n}{n} \\
 &= \frac{n-1}{n} v_i.
 \end{aligned}$$

It is relatively easy to show that this is a symmetric equilibrium bidding strategy. We can apply the Nash equilibrium logic directly. Suppose bidders  $i = 2, \dots, n$  are following the strategy  $b^I(v_i) = \frac{n-1}{n}v_i$  and consider an arbitrary bid  $b$  for bidder 1. Bidder 1 wins with bid  $b$  with probability  $G(b)$ , which is the probability that the other  $n-1$  bidders place bids less than  $b$ . All the other bidders are assumed to bid  $\frac{n-1}{n}v_i$ . Hence the probability that any one of their bids is less than  $b$  is  $\Pr\left[\frac{n-1}{n}v_i < b\right] = \Pr\left[v_i < \frac{n}{n-1}b\right] = \frac{n}{n-1}b$ , in the case of uniform  $F$ . The probability that all of the other  $n-1$  bids is less than  $b$  is, assuming independent value draws,  $\left(\frac{n}{n-1}b\right)^{n-1}$ . Thus, the expected payoff of bidder 1 if she bids an amount  $b$  is

$$G(b)(v_1 - b) = \left(\frac{n}{n-1}b\right)^{n-1} (v_1 - b) = \left(\frac{n}{n-1}\right)^{n-1} b^{n-1}v_1 - b^n.$$

Now let's maximize this with respect to the choice of  $b$ . The first-order condition for a maximum is

$$\left(\frac{n}{n-1}\right)^{n-1} [(n-1)b^{n-2}v_1 - nb^{n-1}] = 0$$

or

$$(n-1)v_1 - nb = 0.$$

The solution is

$$b^* = \frac{n-1}{n}v_1.$$

Hence, if bidder 1 believes that everyone else will follow the symmetric bidding strategy she best responds by following it as well. Since this is true for any bidder, we have verified that  $b^I(v_i) = \frac{n-1}{n}v_i$ , for  $i = 1, \dots, n$ , is a symmetric Nash equilibrium in the case of uniform  $F$ .

*In-class discussion of Experiments 5 and 6: FPA with no entry fee and no reserve for 2- and 5-bidder auctions.*

### 3.2.1 Reserve Prices and Entry Fees

More generally, assume that the first-price auction has an entry fee,  $c$ , and a reserve,  $r$ , where  $0 \leq c \leq 100 - r \leq 100$ . An equilibrium of FPA( $r, c$ ) consists of each bidder following the strategy

$$b^I(v_i) = \begin{cases} \int_{v_0}^{v_i} y \frac{g(y)}{G(v_i)} dy + r \frac{G(v_0)}{G(v_i)} & \text{if } v \geq v_0 \\ \text{No} & \text{if } v < v_0 \end{cases}$$

where  $v_0$  is defined implicitly by the equation  $G(v_0)(v_0 - r) = c$ . We will not go over a formal proof of this claim (See Matthews, 1995, Section 8.1). The value  $v_0$  is the same as in the second-price auction. This is the unique symmetric equilibrium.

**Example 3.1.** *In the case of  $F$  uniform on  $[0, 100]$  and  $n = 2$ ,*

$$b^I(v_i) = \begin{cases} \frac{1}{2}v_i - \frac{v_0^2}{2v_i} + r \frac{v_0}{v_i} & \text{if } v_i \geq v_0 \\ \text{No} & \text{if } v_i < v_0 \end{cases}$$

where  $v_0$  is defined implicitly by the equation  $\frac{v_0}{100}(v_0 - r) = c$ .

*In-class discussion of Experiments 7 and 8: 2-bidder FPA with no entry fee and reserve  $r = \$50$ , 2-bidder FPA with entry fee  $c = \$25$  and no reserve.*

**Useful Information:** For  $r = 0$  and  $c = 25$ , we have  $v_0 = 50$  and hence

$$b^I(v_i) = \begin{cases} \frac{1}{2}v_i - \frac{1250}{v_i} & \text{if } v_i \geq 50 \\ \text{No} & \text{if } v_i < 50. \end{cases}$$

For  $r = 50$  and  $c = 0$ , we have  $v_0 = 50$  and hence

$$b^I(v_i) = \begin{cases} \frac{1}{2}v_i + \frac{1250}{v_i} & \text{if } v_i \geq 50 \\ \text{No} & \text{if } v_i < 50. \end{cases}$$

It is interesting to note that, for the first-price auction, the equilibrium bidding behavior in situations where there is a reserve or an entry fee is different than it is in situations without these features. This is somewhat surprising since we discussed how changing your bidding behavior in the case of the second-price auction if you entered after paying an entry fee is an instance of sunk-cost fallacy. So what's going on? The key is, of course, that you pay your own bid. In the case of a reserve price this means that you might not want to bid a fraction of your value if by doing so you do not meet the reserve. Suppose for example that you have a value of \$60. In a two bidder auction without a reserve you might be tempted to bid \$30. This is the equilibrium strategy in the risk-neutral model. However, if you bid \$30 and the reserve price is \$50, then you will not get the item and your payoff will be zero. In contrast a bid over \$50 but below \$60 will give you a chance at a positive payoff.

In the case of an entry fee, you have to make two decisions. First you have to decide whether or not to enter, and then you have to choose a bid. We already explained how the decision to enter depends on your value. It is important to remember that this is true for all bidders. In the two-bidder first-price auction we considered above, we saw that it was optimal to pay the entry fee and enter the auction if your value was greater or equal than \$50. Since this is true for other bidders as well a person with a value of \$50 should enter the auction and bid \$0! After all, assuming that others are following the same equilibrium bidding strategy as you are, you cannot expect to beat anyone with a higher value than you. Hence, if you have a value of \$50 you are hoping to win only in the event that the other bidder does not enter. Hence, you should enter, but bid \$0.

### 3.2.2 Risk Aversion

In this section we examine the impact of risk aversion on bidding behavior in first-price auctions. For simplicity we will assume there is no entry fee or reserve. We did not discuss risk aversion in the context of second-price auctions because risk aversion does not affect bidding behavior in that auction format when there is no entry fee and no reserve. This is because in a second-price auction a person's bid only affects the probability that they win the auction; it does not affect their payoff. Hence, the decision of how much to bid for any given value does not depend on the person's preferences over payoffs.<sup>2</sup> In the first-price

<sup>2</sup>In the case of a positive entry fee, the minimum value  $v_0$  at which a bidder would be willing to enter the auction would be higher.

auction things are more complicated. Below we will describe the equilibrium bid functions for the case of  $n$  bidders under the assumption that the bidders have constant relative risk aversion (CRRA) utility functions.<sup>3</sup> Again we will only consider the case of no entry fee and no reserve. We will discuss how the theoretical bid functions compare to the experimental classroom data.

We wish to introduce the possibility that bidders exhibit differing degrees of risk aversion. We do so by assuming that each bidder's utility over money payoffs depends on a risk parameter  $\alpha_i$ . In particular, suppose the utility over money payoffs for each bidder  $i$  is given by the function  $u(z) = z^{\alpha_i}$ , where  $0 < \alpha_i < 1$ .<sup>4</sup> This means that bidder  $i$  evaluates a gamble over two different wealth levels in the following way. Her utility of a gamble that gives her a wealth of \$0 with probability .5 and a wealth of \$100 with probability .5 is  $.5u(0) + .5u(100) = .5 \cdot 100^{\alpha_i}$ . If, for example,  $\alpha_i = \frac{1}{2}$ , then  $.5u(0) + .5u(100) = .5 \cdot 100^{1/2} = .5 \cdot 10 = 5$ . In contrast, her utility to receiving the expected value of this gamble,  $.5 \cdot 0 + .5 \cdot 100 = 50$ , is  $u(50) = 50^{1/2}$ , which is approximately equal to 7.07. Hence bidder  $i$ , with risk parameter  $\alpha_i = \frac{1}{2}$ , will prefer to be paid \$50, the expected value of the gamble, for sure to the gamble itself. This means she is risk averse.

In fact, there are a variety of ways to demonstrate that higher degrees of risk aversion are consistent with lower values of  $\alpha_i$  (i.e., closer to 0). For instance, we could contemplate increasing the probability of the \$100 prize in order to get her to accept the gamble rather than a certainty cash payment of \$50. The minimum probability  $p$  that would work is determined by solving

$$p \cdot 100^{\alpha_i} = 50^{\alpha_i}.$$

If  $\alpha_i = \frac{1}{2}$ , the solution is  $p = .707$ . Hence a bidder with risk parameter  $\alpha_i = \frac{1}{2}$ , would accept a gamble that paid \$100 with probability  $p$  to a certainty cash payment of \$50 so long as  $p > .707$ . For general  $\alpha_i$ , the value of  $p$  that solves the above expression is  $p = \left(\frac{1}{2}\right)^{\alpha_i}$ . The lower  $\alpha_i$  is, the higher  $p$  must be. That is, the more risk averse the person is, the more we must compensate her (by giving her favorable terms) to accept a gamble. This will be true for any two money amounts.

Suppose each bidder has a value that is drawn from the uniform distribution on  $[0, 100]$ . I.e.,  $F(v) = v/100$  for all bidders. We will now verify that the symmetric equilibrium bid function is

$$b^I(v_i) = \frac{n-1}{n-1+\alpha_i} v_i,$$

Once again we apply the Nash equilibrium logic directly. Suppose bidder  $i = 2, \dots, n$  follows the strategy  $b^I(v_i) = \frac{n-1}{n-1+\alpha_i} v_i$  and consider an arbitrary bid  $b$  for bidder 1. Bidder 1

<sup>3</sup>See Cox, Smith and Walker, 1982, as a primary source for this material.

<sup>4</sup>This implies constant relative risk aversion equal to  $1 - \alpha_i$ .

wins with bid  $b$  with probability

$$\begin{aligned} \prod_{i=2}^n \Pr \left[ \frac{n-1}{n-1+\alpha_i} v_i < b \right] &= \prod_{i=2}^n \Pr \left[ v_i < \frac{n-1+\alpha_i}{n-1} b \right] \\ &= \frac{1}{100^{n-1}} \prod_{i=2}^n \frac{n-1+\alpha_i}{n-1} b \\ &= \frac{1}{100^{n-1}} \left( \prod_{i=2}^n \frac{n-1+\alpha_i}{n-1} \right) b^{n-1}. \end{aligned}$$

Thus, the expected utility of bidder 1 if she bids an amount  $b$  is

$$\frac{1}{100^{n-1}} \left( \prod_{i=2}^n \frac{n-1+\alpha_i}{n-1} \right) b^{n-1} (v_1 - b)^{\alpha_1}.$$

Now let's maximize this with respect to the choice of  $b$ . The first-order condition for a maximum is

$$\frac{1}{100^{n-1}} \left( \prod_{i=2}^n \frac{n-1+\alpha_i}{n-1} \right) [(n-1)b^{n-2}(v_1 - b)^{\alpha_1} - b^{n-1}\alpha_1(v_1 - b)^{\alpha_1-1}] = 0.$$

Dividing both sides by  $\frac{1}{100^{n-1}} \prod_{i=2}^n \frac{n-1+\alpha_i}{n-1} b^{n-2}(v_1 - b)^{\alpha_1-1}$  yields

$$(n-1)(v_1 - b) - b\alpha_1 = 0$$

or

$$b^* = \frac{n-1}{n-1+\alpha_1} v_1.$$

Hence, if bidder 1 believes that all the other bidders will follow the symmetric bidding strategy she best responds by following it as well.

Note that the equilibrium bid functions are still linear and have an intercept of 0. However, now the slope is somewhere between the risk-neutral prediction of  $\frac{n-1}{n}$  that corresponds to  $\alpha_1 = 1$  and a slope of 1, the slope that corresponds to  $\alpha_1 = 0$ . Note also that each bidder's strategy depends on her own risk-aversion parameter,  $\alpha_i$ , but not on those of the other bidders. Hence, we do not require that individuals know or make any conjectures about the risk aversion of their opponents!

The result that equilibrium bids do not depend on opponents' degree of risk aversion may seem surprising, but it is actually apparent from the risk-neutral case. Specifically, the best response of a bidder in a first price auction involving two bidders is to bid one-half of her value regardless of the fraction between 1/2 and 1 that her opponent bids. In the case of a risk-averse opponent, if we set  $\alpha_1 = 1$  and vary  $\alpha_2$  from 0 to 1 we trace out the same scenarios.

*In-class discussion of individual data from Experiments 5 and 6 using calculations from Assignment 2.*



# Chapter 4

## Revenue

The material in this chapter is motivated by Experiments 1–8.

### 4.1 Expected Revenue

Here we calculate the expected revenue under the efficient equilibrium bidding strategies for the first- and second-price auction formats that were derived formally in Chapter 3.

In a first-price auction with  $F(\cdot)$  uniform on  $[0, 100]$ , the symmetric equilibrium bidding strategy has each bidder bid  $(n - 1)/n$  times their value. Hence, expected revenue is simply  $(n - 1)/n$  times the expected value of the highest value. I.e., Expected revenue is

$$\begin{aligned}\frac{n-1}{n}E[\tilde{v}^{(1)}] &= \frac{n-1}{n} \int_0^{100} v f^{(1)}(v) dv \\ &= \frac{n-1}{n} \int_0^{100} v \frac{nv^{n-1}}{100^n} dv \\ &= (n-1) \int_0^{100} \frac{v^n}{100^n} dv \\ &= \frac{n-1}{100^n} \left[ \frac{v^{n+1}}{n+1} \right]_0^{100} \\ &= \frac{n-1}{n+1} \cdot 100.\end{aligned}$$

In a second-price auction it is a weakly dominant strategy to bid your value, and hence expected revenue is simply the expected value of the second-highest value. In the case where

$F(\cdot)$  is uniform on  $[0, 100]$ ,

$$\begin{aligned}
E[\tilde{v}^{(2)}] &= \int_0^{100} v f^{(2)}(v) dv \\
&= \int_0^{100} v \cdot n(n-1) \left( \frac{v^{n-2}}{100^{n-1}} - \frac{v^{n-1}}{100^n} \right) dv \\
&= n(n-1) \int_0^{100} \left( \frac{v^{n-1}}{100^{n-1}} - \frac{v^n}{100^n} \right) dv \\
&= n(n-1) \left[ \int_0^{100} \frac{v^{n-1}}{100^{n-1}} dv - \int_0^{100} \frac{v^n}{100^n} dv \right] \\
&= n(n-1) \left[ \left[ \frac{v^n}{n \cdot 100^{n-1}} \right]_0^{100} - \left[ \frac{v^{n+1}}{(n+1) \cdot 100^n} \right]_0^{100} \right] \\
&= n(n-1) \left[ \frac{1}{n} - \frac{1}{n+1} \right] \cdot 100 \\
&= n(n-1) \cdot \frac{1}{n(n+1)} \cdot 100 \\
&= \frac{n-1}{n+1} \cdot 100.
\end{aligned}$$

In both auction formats the expected revenue is  $\frac{n-1}{n+1} \cdot 100$ . This is an illustration of a more general revenue equivalence property for standard auctions that will be discussed in more detail in the following section.

## 4.2 Revenue Equivalence

So far, we have been focusing on first- and second-price auctions. For these auction formats we already demonstrated that, in the case where each of the  $n$  bidders' values are generated independently from the uniform distribution on  $[0, 100]$ , the expected revenue to the seller is equal to  $\frac{n-1}{n+1} \cdot 100$ . In fact, the result that the expected revenue is the same in these auction formats is true even if values are generated from a non-uniform distribution. Moreover, revenue equivalence extends to a much broader class of auction formats, which we will refer to as standard auctions.

We now define a *standard auction* to be any auction mechanism in which (i) the object is sold to the person with highest value, and (ii) the expected payment of a bidder with the lowest possible value is 0 in equilibrium. We will assume that all bidders are risk neutral. While we will assume that the bidders' values are independently and identically distributed according to some distribution  $F$ , we will not require that  $F$  be the uniform distribution.

In this auction environment we have the following

**Revenue Equivalence Theorem:** Any symmetric equilibrium of a standard auction  $A$ , involving a strictly increasing bid function yields the same expected revenue to the seller.

*Proof.* Let  $b(\cdot)$  be the symmetric equilibrium bid function and let  $m^A(v)$  denote the expected payment of a bidder with value  $v$  under auction  $A$ . In equilibrium it must be the case that

for any bidder with value  $v$  the bid  $b(v)$  is optimal over all possible bids the bidder could place, given that the other bidders use the bid function  $b(\cdot)$ . Another way to express this is to say that if a bidder contemplated using different values to generate her bid, it would have to be the case that using her true value was optimal. Suppose a bidder contemplates bidding as if her value were  $z$ . She wins the auction if  $b(z) > b(y)$ , where  $y$  is the highest value among the other  $n - 1$  bidders, or equivalently, if  $z > y$ .  $G(\cdot)$  is the distribution function for the random variable  $\tilde{y}$  whose realization is  $y$ . Thus a bidder who reports value  $z$  wins the auction with probability  $G(z)$ . In equilibrium it must be the case that for each bidder

$$\max_z G(z)v - m^A(z)$$

is optimized at  $z = v$ . (Note this is where we use property (i) of the standard auction, the fact that the high bidder wins the auction.)

Hence, the first-order condition is satisfied at  $z = v$ , i.e., we obtain for all  $v$ ,

$$g(v)v - \frac{dm^A(v)}{dv} = 0.$$

Hence,

$$m^A(v) = \int_0^v xg(x) dx + k.$$

At  $v = 0$ ,  $m^A(0) = k = 0$ , by assumption (This is where we use property (ii) of the standard auction, the fact that expected payment of bidder with value 0 is 0.). So,

$$\begin{aligned} m^A(v) &= \int_0^v xg(x) dx \\ &= G(v) \int_0^v x \frac{g(x)}{G(v)} dx \\ &= \Pr[y \leq v] E[y | y < v]. \end{aligned}$$

The right-hand side does not depend on  $A$ . Hence the expected payment of bidders does not depend on  $A$ , and hence the expected revenue to the seller does not depend on  $A$ . Q.E.D.  $\square$

*In-class discussion of revenue estimates from Experiments 1, 2, 5 and 6: 2 and 5-bidder, first and second-price auctions without a reserve or entry fee.*

### 4.3 Revenue With a Reserve Price or Entry Fee

We will address this topic more generally in the next chapter. For now, we will simply compute the expected revenue of the seller in the two cases of second-price auctions involving 2 bidders that we used in the experiments.

Case 1.  $r = \$50$ ,  $c = 0$ .

Case 2.  $r = 0$ ,  $c = \$25$ .

The arguments in the next chapter will show that these expected revenue calculations also apply to the first-price auction.

*In-class discussion of revenue estimates from Experiments 3, 4, 7 and 8: 2-bidder SPA with no entry fee and reserve  $r = \$50$ ; 2-bidder SPA with entry fee  $c = \$25$  and no reserve; 2-bidder FPA with no entry fee and reserve  $r = \$50$ , 2-bidder FPA with entry fee  $c = \$25$  and no reserve.*

# Chapter 5

## Optimal Auctions

The material in this chapter is motivated by Experiment 9.

We wish to analyze the decision of a seller who sets a reserve price when auctioning off an item to a group of  $n$  bidders. We begin by looking at a simpler, but as it turns out, very closely related problem. We consider a seller who faces a single bidder.

We begin with the decision of a seller who chooses an optimal reserve for a second-price auction with one bidder. Clearly the seller who faces a single bidder should set a positive reserve, otherwise the sale price will be zero. In fact, the optimal reserve is equal to the monopoly price. It can be obtained by solving

$$\max_r (1 - F(r))r.$$

The first-order condition is

$$1 - F(r) - f(r)r = 0.$$

So, the optimal reserve (or monopoly price) is defined implicitly as the value of  $r$  that solves

$$r = \frac{1 - F(r)}{f(r)}.$$

We assume that  $F$  is such that this solution is a maximum (i.e., we assume increasing hazard rate).

**Example 5.1.** *Suppose  $F$  is uniformly distributed on the interval  $[0, 100]$ . Then  $r = 50$ .*

Note that if the seller had her own use value for the item,  $v_s$ , then her optimization problem becomes

$$\max_r (1 - F(r))r + F(r)v_s.$$

The first-order condition is

$$1 - F(r) - f(r)r + f(r)v_s = 0.$$

So, the optimal reserve (or monopoly price) is defined implicitly as the value of  $r$  that solves

$$r = \frac{1 - F(r)}{f(r)} + v_s.$$

**Example 5.2.** Suppose  $F$  is uniformly distributed on the interval  $[0, 100]$ . Then  $r = 50 + \frac{v_s}{2}$ .

Consider two scenarios. In scenario 1 the seller sells an item using a first-price auction with no reserve to 2 bidders. In scenario 2 the seller sells an item to a single bidder, but sets the optimal reserve. Assume the bidder(s)' value is drawn from the uniform distribution on  $[0, 100]$ . Compute the expected seller revenue in each case. Discuss.

*Answer:* In scenario 1, the expected revenue to the seller is 33.33 (See Chapter 3). In scenario 2, the optimal reserve is 50, and seller's expected revenue is (since there is only one bidder)

$$.5 \cdot 0 + .5(50) = 25.$$

So the expected revenue in the case where there are two bidders is greater.

There are two striking facts which we will now mention and later verify. Both facts apply to the situation where both the seller and the bidders are risk neutral.

1. The optimal reserve does not depend on the number of bidders.
2. The optimal reserve is the same for the first- and second-price auctions.

Using the above calculations and taking these two facts for granted, one now already knows what the optimal reserve is for both first- and second price auctions, for any number of bidders!

## 5.1 Optimal Reserve with $n \geq 1$ Bidders

Of course, we should never take things for granted. We will now prove that the optimal reserve satisfies the equation

$$r = \frac{1 - F(r)}{f(r)} + v_s$$

in the general case of  $n$  bidders.

First, consider a second-price auction. When there is no reserve, the expected payment of a bidder with value  $v$  is

$$m(v) = \int_0^v y g(y) dy.$$

Why? The bidder bids her value and her payment is the highest value among the other  $n - 1$  bidders, which is distributed according to  $G(\cdot)$ .

If the seller sets a positive reserve price  $r > 0$ , we know that the dominant strategy for each bidder is still to bid  $b(v) = v$ . However, a bidder only wins if  $v > r$ . So, for  $v \geq r$ ,

$$m(v, r) = rG(r) + \int_r^v y g(y) dy.$$

In fact, it follows from the revenue equivalence theorem, that this is the expected payment of a bidder with value  $v$  in the first-price auction with reserve  $r$ .

The seller does not know bidders' values. However, she knows each bidder's value is distributed according to  $F(\cdot)$ . Hence, she can compute the expected payment of a bidder in either the first- or second-price auction

$$\begin{aligned}
E[m(\tilde{v}, r)] &= \int_r^1 m(v, r) f(v) dv \\
&= \int_r^1 rG(r) f(v) dv + \int_r^1 \int_r^v y g(y) dy f(v) dv \\
&= rG(r) \int_r^1 f(v) dv + \int_r^1 \int_y^1 f(v) dv y g(y) dy \\
&= r(1 - F(r))G(r) + \int_r^1 (1 - F(y)) y g(y) dy.
\end{aligned}$$

Note that the third equality is obtained by changing the order of integration. Since this is the expected payment of 1 bidder, the expected payoff to the seller of setting a reserve  $r$  is  $n$  times this amount plus her payoff if she keeps the item. That is, her expected payoff, as a function of  $r$ , is

$$nE[m(\tilde{v}, r)] + F(r)^n v_s = n \left[ r(1 - F(r))G(r) + \int_r^1 (1 - F(y)) y g(y) dy \right] + F(r)^n v_s. \quad (5.1)$$

We can see how this varies with  $r$  by taking the derivative with respect to  $r$ . The result (after some manipulations) is

$$n \left[ 1 - (r - v_s) \frac{f(r)}{1 - F(r)} \right] (1 - F(r))G(r).$$

This shows immediately that it is beneficial to set  $r > v_s$ , since then this derivative is positive. If we assume that  $F$  is such that  $\frac{f(r)}{1 - F(r)}$  is increasing in  $r$  (this is the standard increasing hazard rate assumption) then the maximum expected payoff occurs when the term in the square brackets is zero (up to that point expected payoff increases in  $r$  and after that it declines). Hence, the optimum reserve is given by the value of  $r$  that solves

$$1 - (r - v_s) \frac{f(r)}{1 - F(r)} = 0$$

or

$$r = \frac{1 - F(r)}{f(r)} + v_s.$$

This is the same as the expression we derived for the case of one bidder!

**Example 5.3.** *In the case of  $F$  uniform (where  $G(y) = y^{n-1}$  and  $g(y) = (n-1)y^{n-2}$ ) expected seller revenue as a function of her value and her reserve price choice is, from (5.1), equal to*

$$n \left[ \frac{r^n}{n} - \frac{2r^{n+1}}{n+1} + \frac{n-1}{n(n+1)} \right] + r^n v_s.$$

*In-class discussion of Experiment 9: Reserve choice for 2 and 5 bidder auctions. The formula for computing the optimal reserve values for each treatment is found in Section 6.1 of the textbook. The experiment covers four cases:  $n = 2, v_s = 0 \Rightarrow r = 50$ ;  $n = 2, v_s = 30 \Rightarrow r = 65$ ;  $n = 5, v_s = 0 \Rightarrow r = 50$ ; and  $n = 5, v_s = 30 \Rightarrow r = 65$ . Begin by comparing average observed revenue in each of these four cases to the theoretical predictions. Then examine four plots of 20 rounds of data from the reserve price experiment. Plots are  $n = 2, n = 5$  for  $v_s = 0$ ;  $n = 2, n = 5$  for  $v_s = 30$ ;  $v_s = 0, v_s = 30$  for  $n = 2$ ;  $v_s = 0, v_s = 30$  for  $n = 5$ .*

Consider two scenarios. In scenario 1 the seller sells an item using a first-price auction with no reserve to  $n$  bidders. In scenario 2 the seller sells an item to  $n - 1$  bidders, but sets the optimal reserve. Assume the bidders' values are drawn from the uniform distribution on  $[0, 1]$ . Compute the expected seller revenue in each case. Discuss.

*Answer:* In scenario 1, the expected revenue to the seller is  $\frac{n-1}{n+1}$  (See Chapter 3). In scenario 2, the optimal reserve is .5, and seller's expected revenue is computed as follows. First, the expected payment of any one bidder with value  $v > .5$  is

$$\begin{aligned} .5 \cdot (.5)^{n-1} + \int_{.5}^v y(n-1)y^{n-2} dy &= (.5)^n + (n-1) \int_{.5}^v y^{n-1} dy \\ &= \frac{n-1}{n} v^n + \frac{(.5)^n}{n}. \end{aligned}$$

Hence, a bidder's ex-ante expected payment in the case of  $F$  uniform is

$$\begin{aligned} \int_{.5}^1 \left[ \frac{n-1}{n} v^n + \frac{(.5)^n}{n} \right] dv &= \frac{n-1}{n} \int_{.5}^1 v^n dv + \int_{.5}^1 \frac{(.5)^n}{n} dv \\ &= \frac{n-1}{n} \cdot \frac{1 - (.5)^{n+1}}{n+1} + \frac{(.5)^n}{n} (1 - .5) \\ &= \frac{(.5)^n}{n} - \frac{2(.5)^{n+1}}{n+1} + \frac{n-1}{n(n+1)}. \end{aligned}$$

So, overall, the expected payoff for the seller with value 0 is

$$n \left[ \frac{(.5)^n}{n} - \frac{2(.5)^{n+1}}{n+1} + \frac{n-1}{n(n+1)} \right] = (.5)^n - \frac{n}{n+1} 2(.5)^{n+1} + \frac{n-1}{n+1}.$$

The expression  $(.5)^n - \frac{n}{n+1} 2(.5)^{n+1} > 0$  for all  $n$  and is very close to zero (less than .005) for  $n > 5$ . Hence, for auctions with large numbers of bidders (where here large means more than 5) the seller would be indifferent between having the ability to set a reserve and the addition of one more bidder.

## 5.2 Saliency of the Reserve Price Decision

In our evaluation of the experimental data for the reserve price experiment we might wish to ask whether or not subjects had significant incentive to behave according to the theory. In particular, we conducted 20 rounds of the experiment and we might ask whether subjects' reserve choices improved over time. If the answer is no, it could be because subjects are

not responding to incentives or it could be because the incentives are very small. In what follows, we evaluate how costly it is to make mistakes in the reserve price decision. We do so by computing the expected payoff of the seller as a function of her reserve choice. We present these calculations for each treatment considered in Experiment 9. We also provide the range of reserve price choices that result in a loss of less than 50 cents (or .005 on the 0 to 1 scale) relative to the optimal reserve choice in each case. This information is useful in evaluating the incentives underlying the experimental data.

From the previous section, we know that the expected payoff for the seller with value  $v_s$  and reserve price  $r$  is

$$n \left[ \frac{r^n}{n} - \frac{2r^{n+1}}{n+1} + \frac{n-1}{n(n+1)} \right] + r^n v_s.$$

We now compute expected payoff of the seller as a function of the reserve choice for each of the experimental treatments we conducted in class. Values are multiplied by \$100 to conform with the values used in the experiment.

$n = 2$  and  $v_s = 0$

Seller expected payoff is

$$100 \left( r^2 - \frac{4}{3}r^3 + \frac{1}{3} \right).$$

The seller's expected payoff as a function of the reserve is shown in Figure 5.1.

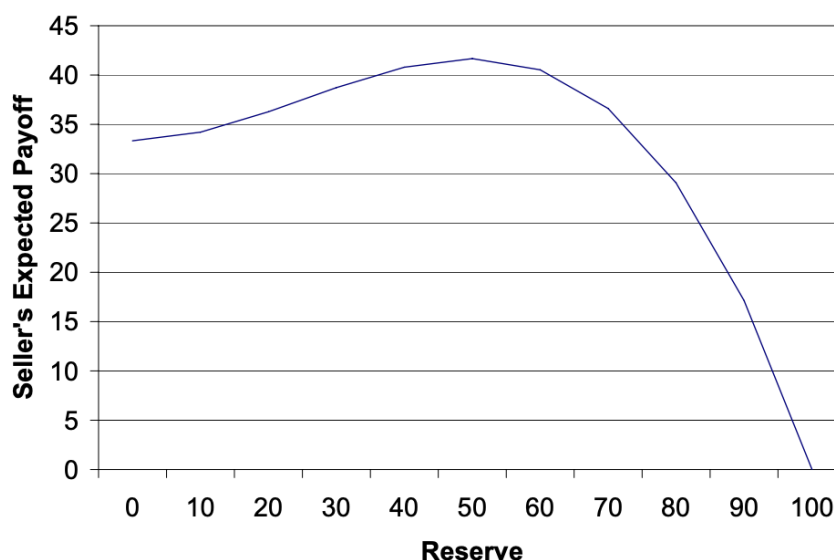


Figure 5.1: The seller's expected payoff as a function of the reserve when  $n = 2$  and  $v_s = 0$ .

For the optimal reserve,  $r = \$50$ , seller expected payoff is \$41.67. Reserve choices that are lower or higher than \$50 cause a significant reduction in expected payoff. The 50 cent optimal bounds are the solution to

$$41.67 - 100 \left( r^2 - \frac{4}{3}r^3 + \frac{1}{3} \right) = .50,$$

i.e.,  $r = \$42.58$  and  $r = \$56.75$ .

$n = 5$  and  $v_s = 0$

Seller expected payoff is

$$100 \left( r^5 - \frac{5}{3} r^6 + \frac{2}{3} \right).$$

The seller's expected payoff as a function of the reserve is shown in Figure 5.2.

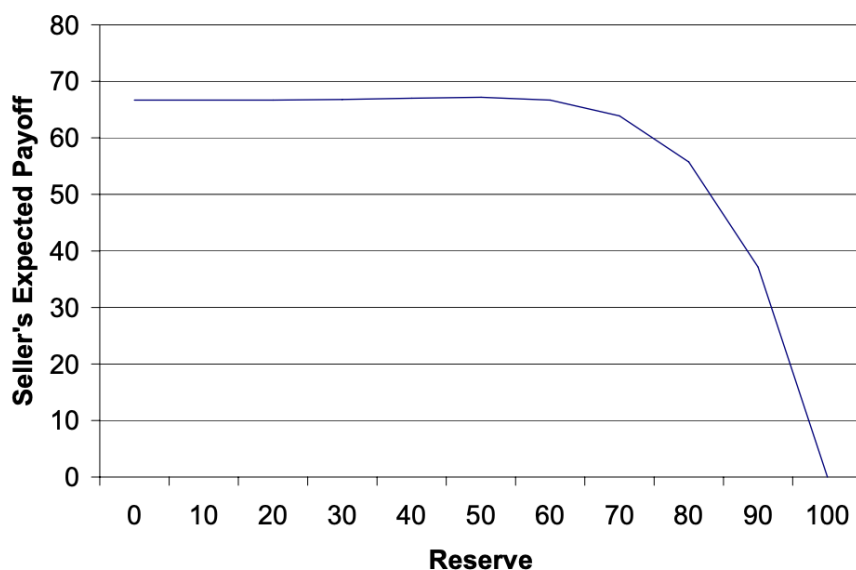


Figure 5.2: The seller's expected payoff as a function of the reserve when  $n = 5$  and  $v_s = 0$ .

For the optimal reserve,  $r = \$65$ , seller expected payoff is \$67.19. Note, that in the case of 5 bidders it is not costly to choose a reserve that is too low. However, reserves that are too high can be quite costly. This is reflected in the 50-cent optimal bounds. The 50 cent optimal bound is the solution to

$$67.19 - 100 \left( r^5 - \frac{5}{3} r^6 + \frac{2}{3} \right) = .50,$$

i.e.,  $r = \$20.40$  and  $r = \$59.82$ .

$n = 2$  and  $v_s = 30$

Seller expected payoff is

$$100 \left( 1.3r^2 - \frac{4}{3} r^3 + \frac{1}{3} \right).$$

The seller's payoff as a function of the reserve is shown in Figure 5.3.

For the optimal reserve,  $r = \$65$ , seller expected payoff is \$51.64. Reserve choices that are lower or higher than \$50 cause a significant reduction in expected payoff. The 50 cent optimal bounds are the solution to

$$51.64 - 100 \left( 1.3r^2 - \frac{4}{3} r^3 + \frac{1}{3} \right) = .50,$$

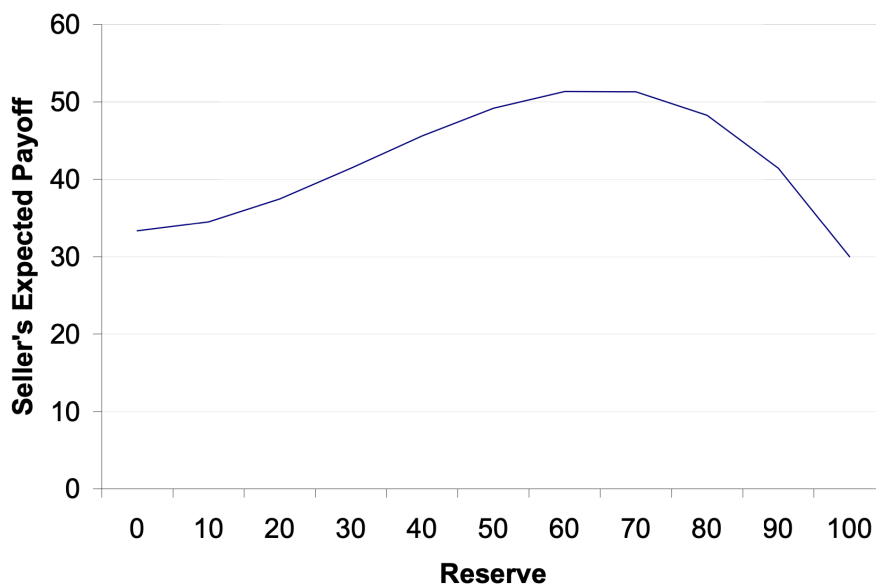


Figure 5.3: The seller's expected payoff as a function of the reserve when  $n = 2$  and  $v_s = 30$ .

i.e.,  $r = \$58.57$  and  $r = \$71.03$ .

$n = 5$  and  $v_s = 30$

Seller expected payoff is

$$100 \left( 1.3r^5 - \frac{5}{3}r^6 + \frac{2}{3} \right).$$

The seller's expected payoff as a function of the reserve is shown in Figure 5.4.

For the optimal reserve,  $r = \$65$ , seller expected payoff is \$69.18. Note, that in the case of 5 bidders it is not costly to choose a reserve that is too low. However, reserves that are too high can be quite costly. This is reflected in the 50-cent optimal bounds. The 50 cent optimal bound is the solution to

$$69.18 - 100 \left( 1.3r^5 - \frac{5}{3}r^6 + \frac{2}{3} \right) = .50,$$

i.e.,  $r = \$55.93$  and  $r = \$71.57$ .

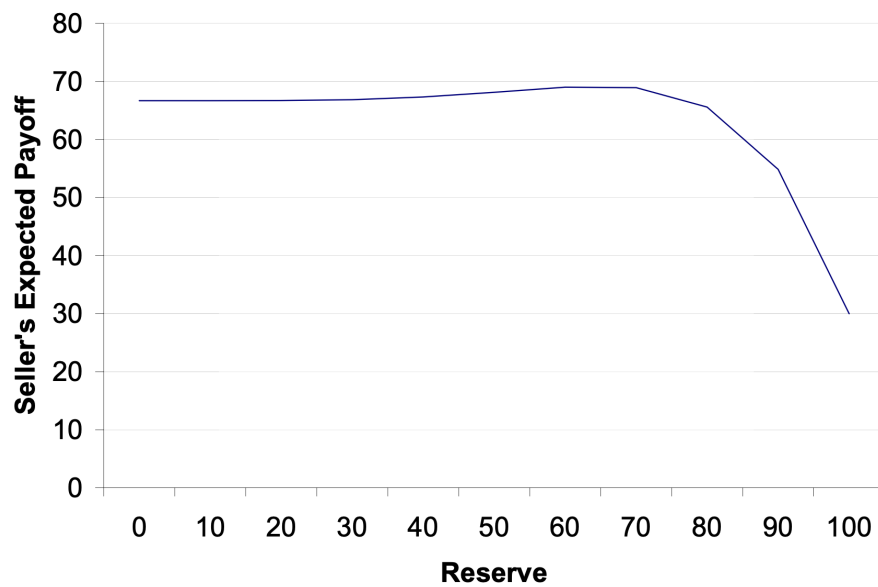


Figure 5.4: The seller's expected payoff as a function of the reserve when  $n = 5$  and  $v_s = 30$ .

*In-class discussion of saliency of Experiment 9.*

# Chapter 6

## Common Value Auctions

The material in this chapter is related to Experiments 10 and 11.

So far we have studied auctions for which bidders have private values. In private value auctions each bidder knows how much she values the item, and this value is her private information. In this chapter we will discuss common value auctions. In common value auctions, the actual value of the item for sale is the same for everyone, but bidders have different private information about what that value is.

Many important auctions are common value auctions. Examples include Treasury bill auctions, auctions of timber, spectrum auctions, and auctions of oil and gas leases. In each case, the value of the item is the same to all the bidders, but different bidders have different information about what that value actually is.

In common value auctions the bidders are often subject to the “winner’s curse.” The winner’s curse has been described in many ways. The simplest definition is the following:

**Winner’s curse:** In a common value auction the bidder with the “best” (most optimistic) information wins. A bidder who fails to take this into account pays, on average, more than the item is worth.

*Example: Penny Jar Experiment*

However, the winner’s curse also results in common value auctions when bidders fail to account for the way private information influences the bidding behavior of their opponents. This idea is best illustrated by the following example.

*Example: Auctioning an Oil Lease*

Player A and Player B are each bidding to purchase the rights to develop an oil field. The field had two parts, Part A and Part B, each of which contains either \$0 or \$3 million worth of oil. Each possibility is equally likely and independently determined. Player A is privately informed about the amount of oil in Part A. Player B is privately informed about the amount of oil in Part B. The two players participate in a first-price auction to purchase the rights to both parts of the field.

We will discuss the results of this experiment in class. Here we derive the theoretical predictions. It is useful to go over the informational aspects of this game before we begin. Player A knows the amount of oil in part A, but does not know the amount of oil in part

B. She only knows that with probability .5 it is \$0 and with probability .5 it is \$3 million. Likewise for player B. How should the players bid in a first-price auction to buy both parts?

No doubt the answer should depend on a bidder's private information. If a bidder sees that her part of the field is worth \$0, she knows the value of both parts is either \$0 or \$3 million. If she sees \$3 million, she knows the value of both parts is either \$3 or \$6 million. Suppose she bids the expected value of the field conditional on her private information. That is, suppose she bids \$1.5 million if she sees \$0, and \$4.5 million if she sees \$3 million. Sound good? Well, it turns out that would be a bad idea. In fact, even if she bids amounts less than these, say to build in a small profit margin, she will most likely be subject to, what economists call, the "winner's curse."

The problem with bidding the expected value of the field conditional on your private information (or even some positive amount less than that) is most evident in the case where the bidder sees \$0. Suppose a bidder sees \$0 and bids \$1 million. This is well below the expected value of the field, which is \$1.5 million. Now ask yourself, how should this bidder feel if she finds out she has won the auction? She would probably be happy at first (we all like to win). But then, she might think, if my bid of \$1 million won, how much oil is likely to be in the field. Note that this is a different question than, how much oil is in the field conditional on my private information. Now we are asking how much is the expected value of the field conditional on the value of my winning bid. This of course depends on the strategy of the opponent and so we can't answer this formally until we introduce the idea of equilibrium strategies. However, it is safe to assume at this point that if the opponent saw \$3 million (and knew there was a 50% chance that the field was worth another \$3 million on top of that) she probably would have bid more than \$1. In other words, if a bidder wins with a bid of \$1 million the field is probably worth \$0! Any bidder who bids a positive amount when she sees \$0 will probably lose money if she wins the auction.

It turns out that a bidder should also bid less than \$4.5 million when she sees \$3 million, but this is hard to explain without conducting the full equilibrium analysis. So, let's do just that. We now verify that the unique symmetric equilibrium of this game is for each bidder to bid according to the bid function:

$$b(0) = 0, \quad b(3) = \tilde{x} \in [0, 3]$$

where  $\tilde{x}$  is distributed according to  $B(x) = \frac{x}{6-x}$ .

Suppose bidder B bids according to the proposed equilibrium strategy. Consider the case where bidder A sees 0, and consider an arbitrary bid  $b_A > 0$  by bidder A. The expected payoff for bidder A is

$$\frac{1}{2}(0 - b_A) + \frac{1}{2}B(b_A)(3 - b_A) = -\frac{1}{2}b_A + \frac{1}{2} \cdot \frac{b_A}{6 - b_A}(3 - b_A) < 0,$$

since  $\frac{3-b_A}{6-b_A} < 1$ . Hence,  $b(0) = 0$  is a best response to player B's strategy.

Next, consider the case where bidder A sees \$3 million. Given the strategy of bidder B, any bid for bidder A between 0 and 3 has an expected payoff of  $\frac{3}{2}$ :

$$\begin{aligned} \frac{1}{2}(3 - b_A) + \frac{1}{2}B(b_A)(6 - b_A) &= \frac{1}{2}(3 - b_A) + \frac{1}{2} \cdot \frac{b_A}{6 - b_A}(6 - b_A) \\ &= \frac{1}{2}(3 - b_A) + \frac{1}{2}b_A = \frac{3}{2}. \end{aligned}$$

Suppose she bids  $b_A > 3$ . Then, her expected payoff is

$$\frac{1}{2}(3 - b_A) + \frac{1}{2}(6 - b_A) = 4\frac{1}{2} - b_A < \frac{3}{2}.$$

Hence,  $b(3) = \tilde{x} \in [0, 3]$ , where  $\tilde{x}$  is distributed according to  $B(x) = \frac{x}{6-x}$ , is a best response to player B's strategy.

We have shown that if player B follows the equilibrium strategy it is a best response for player A to do so as well. If we switch the roles of player A and B, the same argument shows that if player A follows the equilibrium strategy it is a best response for player B to do so as well. That's it. We have verified the Nash equilibrium.

*In-class discussion of Experimental 10: Oil lease experiment.*

Assume that bidders in the oil field experiment can only place whole number bids. Compute the pure strategy Nash equilibrium.

*Answer:* The symmetric Nash equilibrium bid function is  $b(0) = 0$ ,  $b(3) = 2$ .

We will now discuss two, related models of common value auctions. The penny jar example fits the first model, the oil lease example is best captured by the second model.

## 6.1 Model I

The true value of the item being auctioned is  $v$ , but  $v$  is unknown to all bidders. Each bidder  $i$  receives a signal,  $s_i$ , about the true value, which is given by the sum of the true value  $v$  and a random variable  $\tilde{e}_i$ , which you should think of as a private noise term:

$$s_i = v + \tilde{e}_i.$$

We assume that  $\tilde{e}_i$  satisfies  $E[\tilde{e}_i] = 0$  and hence each bidder's signal has the property that  $E[s_i] = v$ . That is, the expected value of each bidder's signal is equal to the true value.

*Reality Check:* Where do these signals come from? The answer depends on the auction environment. For example, in the case of an auction to buy a gold mine, each bidder has to place an estimate on the amount of gold. This estimate can be derived from private tests of rock samples and these tests have errors associated with the predictions. All we are saying in our requirement that  $E[\tilde{e}_i] = 0$ , or equivalently that  $E[s_i] = v$ , is that on average peoples' estimates are correct. In other words, people do not systematically over estimate or underestimate the true value. If they always over estimated it, for example, it would not be surprising that the winner ends up paying too much. Our goal in illustrating the winner's curse is to show that it arises even if peoples' estimates of the true value are correct on average.

We will use this model to illustrate the winner's curse. In particular, we will look at what happens to the winning bidder (in terms of her payoff) if she bids too large a fraction of her signal. We will discuss how this depends on the number of bidders and on how big the noise parameter can be relative to the true value.

For concreteness, we will consider a specific random variable  $\tilde{e}_i$ , that meets our requirement that  $E[\tilde{e}_i] = 0$ . Namely, we assume that each bidder's realized signal  $e_i$  is determined

by a draw from the uniform distribution on  $[-A, A]$ . Hence, the probability that bidder  $i$ 's signal is less than or equal to some value  $e \in [-A, A]$ , is

$$H(e) = \frac{A + e}{2A}.$$

$H(\cdot)$  is the distribution function for the random variable  $\tilde{e}_i$ . It has density function  $h(e) = \frac{1}{2A}$ . Note that  $H(-1) = 0$  and  $H(1) = 1$ , as required. Moreover,

$$E[\tilde{e}_i] = \int_{-A}^A x \cdot \frac{1}{2A} dx = \left[ \frac{x^2}{4A} \right]_{-A}^A = \frac{A}{4} - \frac{A}{4} = 0,$$

as required.

Now suppose all bidders bid a fraction,  $m$ , of their observed signal (We will estimate this fraction using data from a class experiment). We want to show that if  $m$  is too close to 1 the winner loses money in expectation. I say "in expectation" because it is not the case that the winner will always lose money; sometimes improbable things happen and bad decisions can turn out well. However, if a person played the game many times and always bid too close to their value they would lose money.

Since everyone is assumed to bid the same fraction of their signal, the winner will be the bidder with the highest private signal, which is the bidder with the highest realization of the random variable  $\tilde{e}_i$ . We denote the random variable for the highest noise term by  $\tilde{e}^{(1)}$ . In order to compute the winner's expected payoff, we need to compute the expected value of  $\tilde{e}^{(1)}$ . To do this, we first need to know its density function.

Since  $H(e)$  is the distribution of any one noise term (i.e., the realization of any one random variable  $\tilde{e}_i$ ), the probability that all  $n$  realizations of the noise term are less than a value  $e \in [-A, A]$  is

$$H^{(1)}(e) = \left( \frac{A + e}{2A} \right)^n,$$

with density

$$h^{(1)}(e) = n \left( \frac{A + e}{2A} \right)^{n-1} \frac{1}{2A}.$$

Hence

$$E[\tilde{e}^{(1)}] = \int_{-A}^A e \cdot \frac{n}{2A} \left( \frac{A + e}{2A} \right)^{n-1} de = \frac{1}{(2A)^n} \int_{-A}^A ne(A + e)^{n-1} de.$$

Suppose  $n = 2$ . Then

$$\begin{aligned}
 E[\tilde{e}^{(1)}] &= \frac{2}{(2A)^2} \int_{-A}^A e(A+e) de \\
 &= \frac{2}{(2A)^2} \int_{-A}^A (Ae + e^2) de \\
 &= \frac{2}{(2A)^2} \left[ \frac{Ae^2}{2} + \frac{e^3}{3} \right]_{-A}^A \\
 &= \frac{2}{(2A)^2} \left[ \frac{3Ae^2 + 2e^3}{6} \right]_{-A}^A \\
 &= \frac{2}{(2A)^2} \left[ \frac{3A}{6} - \frac{-2A}{6} \right] \\
 &= \frac{A}{3}.
 \end{aligned}$$

Note that even though  $E[\tilde{e}_i] = 0$ , the expectation of  $E[\tilde{e}^{(1)}] > 0$ . Consequently,  $E[\tilde{v}^{(1)}] = v + E[\tilde{e}^{(1)}] > v$ .

Recall that we assume bidders bid some fraction  $m$  of their signal. The implication is that the winner will lose money (in expectation) if

$$m(v + E[\tilde{e}^{(1)}]) - v = m\left(v + \frac{A}{3}\right) - v > 0.$$

This occurs when

$$m > \frac{v}{v + \frac{A}{3}}.$$

For example, suppose that  $v = 3000$  and  $A = 1000$ . Then the winning bidder loses money in expectation only if  $m > \frac{9}{10}$ .

Now let's see what happens if there are more bidders. Does the winner's curse problem get better or worse? Consider  $n = 3$ . With three bidders, the expected value of the winner's signal is greater because the expected value of  $\tilde{e}^{(1)}$  is greater. Namely,

$$\begin{aligned}
 E[\tilde{e}^{(1)}] &= \frac{n}{(2A)^n} \int_{-A}^A e(A+e)^{n-1} de \\
 &= \frac{3}{8A^3} \int_{-A}^A e(A+e)^2 de \\
 &= \frac{A}{2}.
 \end{aligned}$$

Hence,  $E[\tilde{v}^{(1)}] = v + E[\tilde{e}^{(1)}] = v + \frac{A}{2}$ .

Now, assuming all bidders bid a fraction  $m$  of their signal, the winner will lose money (in expectation) if

$$m(v + E[\tilde{e}^{(1)}]) - v = m\left(v + \frac{A}{2}\right) - v > 0,$$

which occurs when

$$m > \frac{v}{v + \frac{A}{2}}.$$

If  $v = 3000$  and  $A = 1000$ , then the winning bidder loses money in expectation if  $m > \frac{6}{7}$ .

Now let's look at what happens when  $n$  gets large. Does the winner's curse get better or worse? Put simply, it gets worse. Intuitively, it should be clear that as  $n$  gets large the expected value of  $\tilde{e}^{(1)}$  will be very close to  $A$ . Think of taking 10,000 random draws of numbers between  $-A$  and  $A$  and ask yourself, "What is the value of the highest draw likely to be?" Hopefully, you said "A." This can also be shown formally. If we further evaluate the integral in the expression for  $E[\tilde{e}^{(1)}]$  we get

$$\begin{aligned} E[\tilde{e}^{(1)}] &= \int_{-A}^A e \cdot \frac{n}{2A} \left( \frac{A+e}{2A} \right)^{n-1} de \\ &= \frac{1}{(2A)^n} \int_{-A}^A ne(A+e)^{n-1} de \\ &= \frac{1}{(2A)^n} \left[ e(A+e)^n \Big|_{-A}^A - \int_{-A}^A (A+e)^n de \right] \\ &= \frac{1}{(2A)^n} \left[ A(2A)^n - 0 - \frac{1}{n+1} (A+e)^{n+1} \Big|_{-A}^A \right] \\ &= \frac{1}{(2A)^n} \left[ A(2A)^n - \frac{1}{n+1} (2A)^{n+1} \right] \\ &= A - \frac{2A}{n+1} \\ &= \frac{n-1}{n+1} A. \end{aligned}$$

Clearly, as  $n \rightarrow \infty$ , we have  $\frac{n-1}{n+1} \rightarrow A$ .

The point is that for large  $n$ ,  $E[\tilde{v}^{(1)}] = v + E[\tilde{e}^{(1)}] = v + A$ . If  $v = 3000$  and  $A = 1000$ , then the winning bidder loses money in expectation if  $m > \frac{3}{4}$ . So even a very cautious bidder, perhaps even one that has been warned about the winner's curse, might lose money!

### General Facts:

1. As the number of bidders increases, a bidder must bid a smaller fraction of her signal to avoid the winner's curse.
2. For any given number of bidders, if the range of the noise parameter is smaller, relative to the true value, the winner's curse results less often. (Not shown)

*In-class discussion of Experiment 11: penny jar experiment.*

## 6.2 Model II

Common values can also be modelled as a special case of interdependent values. In the interdependent values model

$$\begin{aligned}v_1 &= \alpha s_1 + \gamma s_2 \\v_2 &= \alpha s_2 + \gamma s_1\end{aligned}$$

where  $s_1$  and  $s_2$  are private signals of bidders 1 and 2,  $\alpha \geq 0$  is the weight a bidder puts on her own signal and  $\gamma \geq 0$  is the weight she puts on her opponent's signal. We consider the case where  $\alpha = \gamma = 1$ . In this case,  $v_1 = v_2$ , which is a case of common values. Note that the oil lease example fits this model when each signal  $s_i$  is determined independently and  $s_i = 0$  or 3 with equal probability. In what follows we will consider a more general treatment of the private signals.

Suppose that the signals  $s_i$  are drawn independently from the uniform distribution on  $[0, 100]$ .

The first-price auction has a symmetric Nash equilibrium in which each bidder bids  $s_i$ .

*Proof.* Suppose bidder 2 bids  $s_2$  and consider an arbitrary bid  $b_1$  for bidder 1. We need to write down bidder 1's expected payoff as a function of her bid  $b_1$  and show that this is maximized at  $b_1 = s_1$ . We derive bidder 1's expected payoff as a function of her bid in three steps.

**Step 1.** Compute the probability that bidder 1 wins with bid  $b_1$ .

Bidder 1 wins only if her bid is higher than bidder 2's bid, i.e.,  $b_1 > s_2$ . Since we assume that signals are uniform on  $[0, 100]$ , this happens with probability  $\frac{b_1}{100}$ . Thus, bidder 1 wins the auction with bid  $b_1$  with probability  $\frac{b_1}{100}$ .

**Step 2.** Compute bidder 1's expected value of the item if she wins at bid  $b_1$ .

Remember that each bidder's common value for the good is equal to the sum of the private signals. Bidder 1 knows  $s_1$ , but she does not know  $s_2$ . Before the auction begins, the expected value of  $s_2$  is simply 50. But this assumes the random variable  $s_2$  can take on any value between 0 and 100. Bidder 1 is interested in the expected value of  $s_2$  only in the event that she wins the auction with a bid  $b_1$ . As we mentioned in step 1, given the proposed bidding strategy of bidder 2, this only happens if  $s_2 < b_1$ . Hence, we want the expected value of the random variable  $\tilde{s}_2$  conditional on  $s_2 < b_1$ . We know that bidder 2's signal is between 0 and  $b_1$ . Since all these possibilities are equally likely, the expected value of bidder 2's signal is simply half its maximum value, or  $\frac{b_1}{2}$ . Hence, bidder 1's expected value of the item if she wins with bid  $b_1$  is

$$s_1 + \frac{b_1}{2}. \tag{6.1}$$

Let's spend more time making sure you understand what we just did. Suppose bidder 1 wins with a bid of 20. That means the signal of bidder 2 must have been below 20 (because we assume he is bidding his signal and his bid was less than 20). Since we have determined that bidder 2's signal is between 0 and 20, and since all these possibilities are equally likely, the expected value of bidder 2's signal is half its maximum value, or 10, and hence the

expected value of the item for bidder 1 is  $s_1 + 10$ . If, on the other hand bidder 1 were to win with a bid of 90, then bidder 2's signal could be as high as 90. However, the expected value of bidder 2's signal would be half that, or 45, and the expected value of the item for bidder 1 is  $s_1 + 45$ . In both cases, the expected value of the item to bidder 1 is  $s_1 + \frac{b_1}{2}$ , just like in the general formula.

**Step 3.** Compute the expected price bidder 1 pays if she wins with bid  $b_1$ .

Since this is a first-price auction, the price she pays if she wins is her own bid,  $b_1$ .

That's it! We are ready to write down the expected payoff. It is

$$\underbrace{\frac{b_1}{100}}_{\text{Step 1}} \left( \underbrace{s_1 + \frac{b_1}{2}}_{\text{Step 2}} - \underbrace{b_1}_{\text{Step 3}} \right) = \frac{b_1}{100} \left( s_1 - \frac{b_1}{2} \right).$$

To maximize this with respect to the choice of  $b_1$  we take the derivative and set it equal to 0:

$$\frac{s_1}{100} - \frac{b_1}{100} = 0.$$

The solution is  $b_1 = s_1$ , as required.

The same argument can be used to show that  $b_2 = s_2$  is a best response to  $b_1 = s_1$ . That completes the proof.  $\square$

The second-price auction has a symmetric Nash equilibrium in which each bidder bids  $2s_i$ .

*Proof.* Suppose bidder 2 bids  $2s_2$  and consider an arbitrary bid  $b_1$  for bidder 1. We need to write down bidder 1's expected payoff as a function of her bid  $b_1$  and show that this is maximized at  $b_1 = 2s_1$ . We derive bidder 1's expected payoff as a function of her bid in three steps.

**Step 1.** Compute the probability that bidder 1 wins with bid  $b_1$ .

Bidder 1 wins only if her bid is higher than bidder 2's bid, i.e.,  $b_1 > 2s_2$ . Or, equivalently, bidder 1 wins with bid  $b_1$  if  $s_2 < \frac{b_1}{2}$ . Since we assume that signals are uniform on  $[0, 100]$ , this happens with probability  $\frac{b_1}{200}$ . Thus, bidder 1 wins the auction with bid  $b_1$  with probability  $\frac{b_1}{200}$ .

**Step 2.** Compute bidder 1's expected value of the item if she wins at bid  $b_1$ .

Remember that each bidder's common value for the good is equal to the sum of the private signals. Bidder 1 knows  $s_1$ , but she does not know  $s_2$ . She needs to compute the expected value of  $s_2$  conditional on the event that she wins the auction with a bid  $b_1$ , that is, conditional on  $s_2 < \frac{b_1}{2}$ . We know that bidder 2's signal is between 0 and  $\frac{b_1}{2}$ . Since all these possibilities are equally likely, the expected value of bidder 2's signal is simply half its maximum value, or  $\frac{b_1}{4}$ . Hence, bidder 1's expected value of the item if she wins with bid  $b_1$  is

$$s_1 + \frac{b_1}{4}. \tag{6.2}$$

Once again, let's spend a little more time making sure you understand the role bidder 1's winning bid plays in determining her expected value of the item. Suppose bidder 1 wins with a bid of 60. That means the signal of bidder 2 must have been below 30 (because we assume he is bidding twice his signal and his bid was less than 60). Hence, the value of the item, which is the sum of the signals, is at most  $s_1 + 30$ . But, we don't really care about the most it can be, we want to base our decision on its expected value. Since we have determined that bidder 2's signal is between 0 and 30, and since all these possibilities are equally likely, the expected value of bidder 2's signal is half its maximum value, or 15, and hence the expected value of the item for bidder 1 is  $s_1 + 15$ .

**Step 3.** Compute the expected price bidder 1 pays if she wins with bid  $b_1$ .

Since this is a second-price auction, the price she pays if she wins is bidder 2's bid, which is  $2s_2$ . The expected value of bidder 2's bid depends on the bid  $b_1$  because bidder 1 does not win the auction unless  $b_1 > 2s_2$ . We need to compute the expected value of  $2s_2$  conditional on  $b_1 > 2s_2$ . This is done most easily by using a little trick; we use the value  $x = 2s_2$  as the running variable in the integration:

$$E[2\tilde{s}_2 \mid 2s_2 < b_1] = \int_0^{b_1} x \cdot \frac{1}{b_1} dx = \frac{1}{b_1} \left[ \frac{x^2}{2} \right]_0^{b_1} = \frac{b_1}{2}.$$

That's it! We are ready to write down the expected payoff. It is

$$\underbrace{\frac{b_1}{200}}_{\text{Step 1}} \left( \underbrace{s_1 + \frac{b_1}{4}}_{\text{Step 2}} - \underbrace{\frac{b_1}{2}}_{\text{Step 3}} \right) = \frac{b_1}{200} \left( s_1 - \frac{b_1}{4} \right).$$

To maximize this with respect to the choice of  $b_1$  we take the derivative and set it equal to 0:

$$\frac{s_1}{200} - \frac{b_1}{400} = 0.$$

The solution is  $b_1 = 2s_1$ , as required.

The same argument can be used to show that  $b_2 = 2s_2$  is a best response to  $b_1 = 2s_1$ . That completes the proof.  $\square$

The equilibrium we just verified is appealing because it is symmetric, but it is not unique. While we will not prove it, there are other asymmetric equilibria to this auction. Namely, for any  $\lambda > 0$ ,  $b_1(s_1) = (1 + \lambda)s_1$  and  $b_2(s_2) = (1 + \frac{1}{\lambda})s_2$  is a Nash equilibrium. See Osborne's 2004 introduction to game theory text for a proof.

### 6.2.1 Revenue Equivalence

Given these equilibrium bid functions, the expected payment of a bidder is the same in both auction formats. In the first-price auction, bidder  $i$  wins with bid  $s_i$  with probability  $\frac{s_i}{100}$ , and pays  $s_i$ . Hence, bidder  $i$ 's expected payment in the first-price common value auction is  $\frac{s_i^2}{100}$ .

In the second-price auction, bidder  $i$  wins with probability

$$\Pr[2s_i > 2\tilde{s}_j] = \Pr[s_i > \tilde{s}_j] = \frac{s_i}{100}$$

and she pays

$$E[2\tilde{s}_j \mid s_j < s_i] = \int_0^{s_i} 2s_j \cdot \frac{1}{s_i} ds_j = \frac{2}{s_i} \left[ \frac{x^2}{2} \right]_0^{s_i} = s_i.$$

Hence, bidder  $i$ 's expected payment in the second-price common value auction is  $\frac{s_i^2}{100}$ .

So, in both the first and second-price, common-value auction expected revenue is

$$2E\left[\frac{\tilde{s}_i^2}{100}\right] = 2 \int_0^{100} \frac{s_i^2}{100} \cdot \frac{1}{100} ds_i = 2 \left[ \frac{s_i^3}{30000} \right]_0^{100} = \frac{200}{3}.$$

Revenue equivalence holds in this setting!